ACC For Log Canonical Threshold Polytopes

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Preliminaries

Log canonical threshold polytope (LCT-polytope). Let (X,Δ) be a lc pair and $D_1,\ldots,D_s$ be effective $\mathbb{R}$-Cartier divisors. The LCT-polytope $P(X,\Delta;D_1,\ldots,D_s)$ of $D_1,\ldots,D_s$ is

$$\{(t_1,\ldots,t_s)\in\mathbb{R}^s_{\geq 0}\mid (X,\Delta+t_1D_1+\ldots+t_sD_s) \text{ is log canonical}\}.$$

Example. If $s=1$, $P(X,\Delta;D)$ is just the interval $[0,\operatorname{lct}(X,\Delta;D)]$, where $\operatorname{lct}(X,\Delta;D)$ is the log canonical threshold of D.

DCC and ACC set. For a partially ordered set, DCC refers to descending chain condition, and ACC refers to ascending chain condition. For sequence of polytopes, we choose inclusion "$\subseteq$" as the partial order.

Main Theorem

ACC for LCT-polytopes

Let $n,s\in\mathbb{N}$ be fixed natural numbers and $\mathcal{I}$ be a DCC set of positive real numbers. We assume that $\mathcal{S}=\{(X,\Delta;D_1,\ldots,D_s)\}$ is a set whose elements satisfy the following properties:

- X is a normal variety of dimension n,
- (X,Δ) has log canonical singularities with coefficients of Δ belong to $\mathcal{I}$, and
- $D_1,\ldots,D_s$ are $\mathbb{R}$-Cartier divisors, and the coefficients of $D_1,\ldots,D_s$ belong to $\mathcal{I}$.

Then, the set of LCT-polytopes

$$\{P(X,\Delta;D_1,\ldots,D_s)\mid (X;\Delta;D_1,\ldots,D_s)\in\mathcal{S}\}$$

is an ACC set under the inclusion.

Remarks.

- ACC for log canonical thresholds (i.e. $s=1$) is proved in [HMX14]. We rely on this result.
- LCT-polytopes is defined in [LM11]. The above result is proven under the smoothness assumption on X by generic limit method.

Idea of proof

For two divisors ($s=2$).

Let $P_i := P(X_i;D_{i,1},D_{i,2})\subseteq\mathbb{R}^2_{\geq 0}$ be an increasing sequence of LCT-polytopes. Intersect each P_i by a vertical line $\{x=p\}$. The intersection point $(p,t_i(p))$ is determined by

$$t_i(p)=\sup\{t\mid (X_i,pD_{i,1}+tD_{i,2}) \text{ is log canonical}\}.$$

Thus, $t_i(p)$ is the log canonical threshold of $D_{i,2}$ for $(X_i,pD_{i,1})$. The coefficients of $D_{i,2},pD_{i,1}$ lie in the DCC set $\mathcal{I}\cup p\mathcal{I}$. By ACC for log canonical thresholds, $\{t_i(p)\}_{i\in\mathbb{N}}$ stabilizes.

However, by knowing $\{P_i\}_{i\in\mathbb{N}}$ stabilizes along any vertical line is not enough to conclude that LCT-polytopes themselves stabilize. For example, it could happen that there is a fixed vertex τ for all $P_i, i\gg 1$, but the 1-dimensional faces connecting with this vertex do not coincide (see Figure 1).

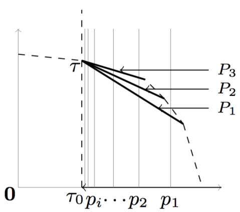

Figure 1: A sequence of strictly increasing polytopes

It turns out that Figure 1 is the only configuration we need to worry about. By the birational geometry technique, we can show that such configuration could never happen.

Generalizes above to more divisors.

- Find the higher dimensional analogous to "Figure 1". Show this is the only obstruction to ACC. This step only needs elementary properties of polytopes.
- Show the above obstruction cannot happen. This step uses birational geometry.

Other results

Let D_i be a prime divisor, and $f_i(t)$ be a real function of t. Consider the divisor $\Delta(t)=\Sigma_i f_i(t)D_i$. If all the $f_i(t)$ are linear functions, we have global ACC generalizes the case of real coefficients in [HMX14].

Global ACC for linear coefficients

Let $n\in\mathbb{N}$ and $a,b\in\mathbb{R}_{>0}$ be fixed numbers. Let $\mathcal{F}$ be a set of real linear functions and $\{(X,\Delta(t))\}$ be a set of log pairs. Suppose they satisfy the following properties:

- X is normal with $\dim X=n$,
- for any $f(t)\in\mathcal{F}$, $f(t)\geq 0$ on $[a,b]$, and $\mathcal{F}|_a,\mathcal{F}|_b$ are both DCC sets, where for $c\in[a,b]$, $\mathcal{F}|_c := \{f(c)\mid f(t)\in\mathcal{F}\}$,
- the coefficients of $\Delta(t)$ are in $\mathcal{F}$,
- there exists $a<b_X\leq b$, such that $(X,\Delta(t))$ is lc on $[a,b_X]$, and
- $K_X+\Delta(t)\equiv 0$ on $[a,b]$.

Then there is a *finite* set $\mathcal{F}'$ such that the coefficients of $\Delta(t)$ are in $\mathcal{F}'$ for each $\{(X,\Delta(t))\}$.

Similarly, one can obtain the result of ACC for Fano spectrum in the case of linear coefficients.

Question: Accumulations of LCT-polytopes

It is known that the accumulation points of log canonical thresholds lie in the set of log canonical thresholds of lower dimensional varieties. It is not known whether the similar property holds for LCT-polytopes, even for smooth varieties. Because our method is local, we are unable to deal with this global property. On the other hand, [LM11] shows that in the smooth case, the LCT-polytopes converge to LCT-polytope (of the same dimensional varieties) in the Hausdorff metric by generic limit method.

Potential applications

LCT-polytopes might be applied to the problems on the existence of Kähler-Einstein metrics. Analytically, α-invariant is introduced to deal with such problem. α-invariant is the log canonical thresholds of some $\mathbb{R}$-linear system. Log canonical threshold also appeared in the study of stabilities of varieties. It is desirable to see if LCT-polytopes could give some refined measurement for the existence of Kähler-Einstein metrics.

References

[HMX14] C. Hacon, J. McKernan, and C. Xu. ACC for log canonical thresholds. *Ann. of Math. (2)*, 180(2):523–571, 2014.

[LM11] A. Libgober and M. Mustaţă. Sequences of LCT-polytopes. *Math. Res. Lett.*, 18(4):733–746, 2011.